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ON THE BRAUER GROUP OF A PROJECTIVE VARIETY

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ABSTRACT

This paper presents a direct, torsion-theoretic description of the Brauer group of a projective scheme X. If X is a regular projective variety of dimension at most two, then $\mathrm{Br}(X)$ reduces to the relative Brauer group of the homogeneous coordinate ring of X, based on pseudo Azumaya algebras.

0. Introduction

In [32] the first author introduced the so-called "graded" Brauer group of a graded commutative ring C. It is defined in terms of graded Azumaya algebras over C and contains an amount of information on the graded arithmetical structure of C. In particular, its relation to the common Brauer group in case C is an arithmetically graded ring has been studied in loc. cit. This allows us to apply these techniques in case C is the homogeneous coordinate ring of a nonsingular projective plane curve. Let us point out that the reason why the theory of arithmetically graded rings may be brought to bear on the geometrical situation is that the graded ring of quotients of C at a graded prime ideal P in $\mathrm{Proj}(C)$, say $Q_P^g(C)$, is a generalized Rees ring in the sense of [31] such that its part of degree zero $Q_P^g(C)_0 = C_{(P)}$ (the stalk at P of the structure sheaf on the curve) is a discrete valuation ring.

Now, A. Grothendieck introduced and studied the Brauer group of a scheme in [12], but, although essentially the same for schemes, we prefer to introduce this notion through B. Auslander's definition of the Brauer group of an arbitrary ringed space, cf. [3]. According to loc. cit. we thus have to consider locally separable sheaves of algebras over an arbitrary ringed space. It appears that the Brauer group of an affine scheme $\mathrm{Spec}(R)$ coincides with the usual Brauer Group of the ring R, in the sense of M. Auslander and O. Goldman, cf. [5].

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The aim of this paper is to study the Brauer group of Proj(R) for a graded commutative ring R. Although the results of the first sections will be shown to hold for arbitrary rings R, we will be led in a most natural way to restrict attention to regular, projective varieties over an arbitrary, not necessarily algebraically closed field. Note that the present paper is a revised version of [40], where only the normally projective curve case was dealt with. One of the techniques which make our methods work is the use of idempotent kernel functors, a technique which seems to be missing in most geometrical considerations. Indeed, if R_+ is the graded ideal of R generated by all homogeneous elements of strictly positive degree and if $\sigma^g_{R_+}$ is the graded kernel functor associated with the idempotent filter generated by the powers of the ideal R_+, then, in a sense, the Brauer group of Proj(R) is the Brauer group of R in the category of graded $\sigma^g_{R_+}$-closed R-modules. A similar property holds for the Brauer group of a not necessarily affine subscheme of an affine scheme Spec(R). That this is not as far fetched as it seems may be seen in case R is the homogeneous coordinate ring of some projective variety. Actually it turns out that the graded localization of R at $\sigma^g_{R_+}$ reduces to the integral closure of R in its field of fractions. The main result of this paper deals with regular projective varieties. As a consequence of this result we obtain the interesting corollary that for regular projective varieties of dimension at most two the Brauer group of the variety coincides with the relative graded Brauer group of its homogeneous coordinate ring, cf. [41].

Finally, both authors would like to express their thanks to Ray Hoobler, some of whose remarks enabled them to produce this revised version of [40].

1. Some sheaf theoretical background

1.1. In this section we will recall some definitions and results, contained in [3]. Proofs are omitted, for full information we refer to loc. cit. Throughout $(X, \boldsymbol{O}_X)$ will denote a commutatively ringed space, i.e. X is a topological space and $\boldsymbol{O}_X$ is a sheaf of commutative rings on X. The category of sheaves of $\boldsymbol{O}_X$ modules will be denoted by $\boldsymbol{O}_X$-Mod. A sheaf of $\boldsymbol{O}_X$-modules $\boldsymbol{M}$ is said to be locally projective of finite type if $\boldsymbol{M}$ is locally a direct summand of a free sheaf of $\boldsymbol{O}_X$-modules of finite rank, i.e. for every $x \in X$ there exists an open neighborhood U of x and morphisms $\psi : \boldsymbol{M} \mid U \to \boldsymbol{O}_X^n \mid U$ resp. $\varphi : \boldsymbol{O}_X^n \mid U \to \boldsymbol{M} \mid U$ such that $\varphi \circ \psi$ is the identity on $\boldsymbol{M} \mid U$. Here, as usually, $\boldsymbol{M} \mid U$ denotes the restriction of $\boldsymbol{M}$ to U. From [3] we recall

(1.2) PROPOSITION. *For any $M \in O_X$-Mod the following statements are equivalent:*

(1.2.1) *M is locally projective of finite type;*

(1.2.2) *M is locally of finite type and for all open subsets U of X the functor* $\mathrm{Hom}_{O_X|U}(M \mid U, -)$ *is exact in* $O_X \mid U$-Mod;

(1.2.3) *M is finitely presented and for all $x \in X$ the $O_{X,x}$-module M_x is projective.*

□

(1.3) The full subcategory of O_X-Mod, consisting of all locally projective sheaves of finite type, possesses several nice stability properties. Indeed, if M and N are such sheaves of O_X modules, then so are $\mathbf{Hom}_{O_X}(M, N)$ and $M \otimes_{O_X} N$. Moreover, there are then canonical isomorphisms

$$\mathbf{Hom}_{O_X}(M, M) \otimes_{O_X} \mathbf{Hom}_{O_X}(N, N) \xrightarrow{\sim} \mathbf{End}_{O_X}(M \otimes_{O_X} N)$$

and

$$\mathbf{Hom}_{O_X}(M, O_X) \otimes_{O_X} M \xrightarrow{\sim} \mathbf{Hom}_{O_X}(M, M).$$

Note also that if M is locally projective of finite type and if N is any presheaf of O_X-modules then for every $x \in X$ we may find an open neighbourhood U such that the canonical morphism $M \otimes'_{O_X} N \mid U \to M \otimes_{O_X} N \mid U$ is an isomorphism of presheaves (where $\otimes'$ denotes the tensor-product of presheaves!).

(1.4) Let us define a natural morphism of sheaves of rings $\omega : O_X \to \mathrm{Hom}_{O_X}(M, M)$ by mapping "α in O_X to the left multiplication by α" — this "abus de langage" should be understood locally. The kernel of ω is the annihilator $\mathbf{Ann}_{O_X}(M)$ of M; if $\mathbf{Ann}_{O_X}(M) = 0$ then we speak of a faithful sheaf of O_X-modules. If M is finitely presented, then M is faithful if M_x is a faithful $O_{X,x}$-module for every $x \in X$. Let us denote by W the sheaf $\mathbf{Hom}_{O_X}(M, M)$, then M is a sheaf of W-modules in the obvious way. If M is faithful, locally projective and of finite type when viewed as a sheaf of O_X-modules, then it has the same properties as a sheaf of W-modules. Note also that if M and N_1 have the foregoing properties and if N_2 is an arbitrary sheaf of O_X-modules, together with an isomorphism $N_2 \otimes_{O_X} M \xrightarrow{\sim} N_1$, then N_2 is faithful and locally projective of finite type.

(1.5) PROPOSITION. *Let (X, O_X) be a scheme and M a sheaf of O_X-modules, then the following properties are equivalent:*

(1.5.1) *M is locally projective of finite type;*

(1.5.2) *M is locally free of finite rank.*

PROOF. One proves this by reducing to the case of affine schemes Spec(R), when one proves that locally projective sheaves of finite type correspond to finitely generated projective R-modules, cf. [3]. □

(1.6) To any sheaf of O_X-algebras L one associates L^0 by putting $L^0(U) = (L(U))^0$ for any open subset U of X; the enveloping algebra will be $L^e = L \otimes_O L^0$. Clearly L may be considered as a sheaf of left L^e-modules and it is easily verified that if L is of finite type in O_X-Mod, then L is finitely presented in L^e-Mod.

We have the following diagram of monomorphisms:

$$\begin{array}{ccc} O_X & \xrightarrow{\ \gamma\ } & \mathbf{Hom}_{L^e}(L, L) \\ & {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & L & \end{array}$$

where i is the structural morphism for L, γ is defined locally by associating to an element the right multiplication by that element, and β is just the evaluation at the identity of L. One usually identifies $\mathbf{Hom}_{L^e}(L, L)$ with its image $Z(L)$ in L. We refer to $Z(L)$ as the center of L; whenever γ is an isomorphism, we say that L is a *central sheaf* of O_X-algebras or simply that L is *central* over O_X. It is clear that a finitely presented L in O_X-Mod is central over O_X if and only if L_x is a central $O_{X,x}$-algebra for every $x \in X$.

(1.7) Let us define a homomorphism $\eta : L^e \to \mathbf{Hom}_{O_X}(L, L)$ of sheaves of rings "locally by associating to $a \otimes b$ the morphism $x \to axb$. A central sheaf of O_X-algebras L which is locally projective of finite type over O_X is said to be a *locally separable* sheaf of O_X-algebras if η is an isomorphism. Equivalently a sheaf of O_X-algebras L is locally separable if and only if L is finitely presented in O_X-Mod and L_x is a central separable $O_{X,x}$-algebra for each $x \in X$. We may refer to a locally separable sheaf of O_X algebras over a scheme X as an Azumaya Algebra (in the sense of A. Grothendieck's [12]) in view of (1.5). If M is faithful, locally projective of finite type, then $\mathbf{Hom}_{O_X}(M, M)$ is locally separable. Moreover, if L_1 and L_2 are locally separable, then so is $L_1 \otimes_{O_X} L_2$. Finally if L is locally separable over O_X and O'_X is a commutative sheaf of O_X- algebras, then $L \otimes_{O_X} O'_X$ is locally separable over O'_X.

(1.8) Let $C_1(X, O_X)$ be the category of locally separable sheaves of central O_X-algebras with O_X-algebra morphisms. We have seen that $C_1(X, O_X)$ is closed under tensor-products over O_X. Let $C'_1(X, O_X)$ be the full subcategory of

$C_1(X, O_X)$ consisting of those objects isomorphic as sheaves of algebras to sheaves of the form $\mathbf{Hom}_{O_X}(M, M)$ for some faithful locally projective sheaf M of finite type. From the above it is clear that $C_1'(X, O_X)$ is closed under tensor-products too. Let us now say that L_1 and L_2 in $C_1(X, O_X)$ are equivalent ($L_1 \sim L_2$) if and only if there exist $W_1, W_2 \in C_1(X, O_X)$ such that $L_1 \otimes_{O_X} W_1 \xrightarrow{\sim} L_2 \otimes_{O_X} W_2$. The set of equivalence classes $B(X, O_X)$ for this relation may be endowed with the structure of an abelian group through the multiplication induced by taking tensor-products over O_X, such that the class of L^0 is the inverse of the class of L and where the class of O_X is the unit element. Obviously for any $L \in C_1(X, O_X)$ we have $L \in C_1'(X, O_X)$ if and only if $L \sim O_X$. The group $B(X, O_X)$ is called the *Brauer group* of the ringed space (X, O_X).

(1.9) PROPOSITION. *Let* Spec(R) *be an affine scheme with structure sheaf* O_R *and let* Br(R) *be the Brauer group of* R *in the sense of M. Auslander and O. Goldman, then there is a canonical isomorphism* $B(\mathrm{Spec}(R), O_R) \xrightarrow{\sim} \mathrm{Br}(R)$. □

The purpose of this paper is to provide a similar description of the Brauer group of a projective scheme, at least in some special cases.

2. The projective case

(2.1) Throughout $R = \bigoplus_{n=0}^{\infty} R_n$ denotes a commutative, positively graded ring with unit. Put $\mathrm{Proj}(R) = \{P$ graded prime ideal of R; $P \not\supset R_+\}$, where $R_+ = \bigoplus_{n=1}^{\infty} R_n$. To any graded ideal I of R associate $V_+(I) = \{P \in \mathrm{Proj}(R)$; $I \not\subset P\}$, and the Zariski topology on Proj(R) is defined by taking the $X_+(I) = \mathrm{Proj}(R) - V_+(I)$ as open subsets. If $Q_I^g(R)$ is the graded ring obtained by localizing at the graded torsion theory described by the filter of ideals generated by the powers of I (cf. [21], [34], details on graded localization), then we may construct a sheaf of graded algebras O_R^+ on Proj(R) by putting $\Gamma(X_+(I), O_R^+) = Q_I^g(R)$. The stalk of O_R^+ at $P \in \mathrm{Proj}(R)$ is given by $O_{R,P}^+ = Q_{R-P}^g(R)$ (cf. [14]) where $Q_{R-P}^g(R)$ is obtained from R by inverting the homogeneous elements in $R - P$, i.e. in the multiplicative system $h(R - P)$. We call O_R^+ the *graded structure sheaf* on Proj(R). The *structure sheaf* on Proj(R) is then just $\tilde{R} = (O_R^+)_0$ defined by $\Gamma(X_+(I), \tilde{R}) = (Q_I^g(R))_0$, the part of degree zero of $Q_I^g(R)$. In a similar way one constructs O_M^+ and $\tilde{M}$ for arbitrary graded R-modules M.

(2.2) Recall that for any integer n one defines the shift functor T_n by its action on an R-module M given by $(T_nM)_p = M_{n+p}$ for all $p \in \mathbf{Z}$. Let us denote by $O_R^+(n)$ resp. $\tilde{R}(n)$ the sheaf of O_R^+-modules resp. $\tilde{R}$-modules associated with T_nR in the above way. If M is a sheaf of $\tilde{R}$-modules, then we write $M(n)$ for

$\boldsymbol{M}\otimes_{\tilde{R}}\tilde{R}(n)$. We define a left exact functor Γ_* from sheaves of $\tilde{R}$-modules to graded R-modules by putting $\Gamma_*(\boldsymbol{M})=\bigoplus_n \Gamma(\mathrm{Proj}(R),\boldsymbol{M}(n))$. Recall further that for any pair of graded R-modules M, N we define $\mathrm{HOM}_R(M,N)$ to be the graded R-module generated by the graded R-linear morphisms $f:M\to N$ of arbitrary degree p, i.e. having the property that $f(M_m)\subset N_{m+p}$ for any $m\in\mathbf{Z}$. The category R-gr has as objects graded R-modules and as morphisms graded R-linear morphisms of degree 0, i.e. which are degree-preserving. So $\mathrm{Hom}_{R\text{-gr}}(M,N)=\mathrm{HOM}_R(M,N)_0$. Assume from now on that R *is generated as an* R_0*-algebra by a finite number of elements of degree* 1. In geometrical applications we will even have that R_0 is a field and R is noetherian. Let us recall some well-known facts in the following

(2.3) PROPOSITION.

(2.3.1) *For any quasicoherent sheaf of $\tilde{R}$-modules the canonical morphism* $\beta:(\Gamma_*(\boldsymbol{M}))^{\sim}\to\boldsymbol{M}$ *is an isomorphism;*

(2.3.2) *every quasicoherent sheaf of $\tilde{R}$-modules (of finite type) $\boldsymbol{M}$ is of the form $\tilde{M}$ for some (finitely generated) graded R-module M;*

(2.3.3) *if M and N are graded R-modules, with M finitely presented, then there is a canonical isomorphism* $\mu:(\mathrm{HOM}_R(M,N))^{\sim}\to\mathbf{Hom}_{\tilde{R}}(\tilde{M},\tilde{N})$:

(2.3.4) *for any pair of graded R-modules M, N there is a canonical isomorphism* $\lambda:\tilde{M}\otimes_{\tilde{R}}\tilde{N}\to(M\otimes_R N)^{\sim}$;

(2.3.5) *if M is a graded R-module then $\tilde{M}=0$ if and only if for all $m\in M$ and $r\in R_n$, $n>1$, we may find $p\in\mathbf{N}$ such that $r^pm=0$;*

(2.3.6) *for any graded R-module M we have $(T_nM)^{\sim}=\tilde{M}(n)$;*

(2.3.7) *let R_0 be a finitely generated algebra over a field and let M be a finitely generated graded R-module; for large enough positive integers $d\in\mathbf{N}$ the canonical map $\alpha:M\to\Gamma_*(\tilde{M})$ induces an isomorphism $\alpha_d:M_d\to\Gamma(X,\tilde{M}(d))$.*

(2.4) LEMMA. *Let S be a commutative graded ring such that $SS_1=S$. The Grothendieck categories S-gr and S_0-mod are naturally equivalent and this equivalence is given by the functors $(-)_0:S\text{-gr}\to S_0\text{-mod}$, $M\mapsto M_0$, and $S_{S_0}\otimes -:S_0\text{-mod}\to S\text{-gr}$, $N\mapsto S_{S_0}\otimes N$.*

PROOF. First note that the condition $SS_1=S$ entails $S_nM_m=M_{n+m}$ for every $M\in R$-gr, $n,m\in\mathbf{Z}$. Consider the graded module $S_{S_0}\otimes S_n$, for some $n\in\mathbf{Z}$, the graduation being given by $(S_{S_0}\otimes S_n)_m=S_{m,S_0}\otimes S_n$. The map $\psi:S_{S_0}\otimes S_1\to S(n)$ given by $s\otimes r_n\mapsto sr_n$ is graded of degree zero. Moreover ψ is surjective because $S(n)$ is as a graded left S-module generated by $S(n)_0=R_n$. Now $\mathrm{Ker}\,\psi$ is a graded S-module, i.e. generated by its part of degree zero. The exact sequence in S-gr:

$$0 \to \operatorname{Ker} \psi \to S_{S_0}\otimes S_n \to S(n) \to 0$$

yields an exact sequence in S_0-mod:

$$0 \to (\operatorname{Ker} \psi)_0 \to S_n \to S_n \to 0$$

by taking parts of degree zero. Consequently: $(\operatorname{Ker}\psi)_0 = 0$ and thus $\operatorname{Ker}\psi = 0$. Therefore $S_{S_0}\otimes S_n \cong S(n)$ in S-gr and thus $S_{m\,S_0}\otimes S_n \cong S_{n+m}$ in S_0-mod. From $S_{-n\,S_0}\otimes S_n \cong S_0 \cong S_{n\,S_0}\otimes S_{-n}$ it follows that S_n is an invertible S_0-module hence finitely generated projective. This entails, e.g., that S is flat as an S_0-module. Finally, for every $M \in S$-gr we have an exact sequence in S-gr:

$$0 \to K \to M_{0\,S_0}\otimes S \to M \to 0,$$

and as before we deduce that $K_0 = 0$, hence $K = 0$. The statements of the lemma are now easily verified.

(2.5) Proposition. *Let M and N be graded R-modules with M finitely presented; if we write E for $\mathrm{HOM}_R(M, N)$, then there is a canonical isomorphism $\mu : \boldsymbol{O}_E^+ \to \mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{O}_M^+, \boldsymbol{O}_N^+)$.*

Proof. Recall that $\mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{O}_M^+, \boldsymbol{O}_N^+)$ is defined by putting for each open subset U of X

$$\Gamma(U, \mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{O}_M^+, \boldsymbol{O}_N^+)) = \mathrm{HOM}_{\boldsymbol{O}_R^+|U}(\boldsymbol{O}_M^+ | U, \boldsymbol{O}_N^+ | U),$$

where, again, for arbitrary sheaves of graded modules $\boldsymbol{M}$, $\boldsymbol{N}$ over a sheaf of graded rings $\boldsymbol{R}$ we define $\mathrm{HOM}_{\boldsymbol{R}}(\boldsymbol{M}, \boldsymbol{N})$ to consist of all "compatible" families $\{f_U : U \text{ open in } X\}$ where $f_U \in \mathrm{HOM}_{\boldsymbol{R}(U)}(\boldsymbol{M}(U), \boldsymbol{N}(U))$.

Now, if $f \in R_d$, we define a morphism $\mu_{f,p} : Q_f^g(\mathrm{HOM}_R(M,N)_p \to (\mathbf{HOM}_{Q_f^g(R)}(Q_f^g(M), Q_f^g(N))_p : u/f^n \to (x/f^r \to u(x)/f^{r+n})$, where $u \in \mathrm{HOM}_R(M, N)_{n+p}$. If $e \in R_d$, then we obtain a commutative diagram:

$$\begin{array}{ccc} Q_f^g(\mathrm{HOM}_R(M,N)) & \xrightarrow{\mu_f} & \mathrm{HOM}_{Q_f^g(R)}(Q_f^g(M), Q_f^g(N)) \\ \downarrow & & \downarrow \\ Q_{ef}^g(\mathrm{HOM}_R(M,N)) & \xrightarrow[\mu_{ef}]{} & \mathrm{HOM}_{Q_{ef}^g(R)}(Q_{ef}^g(M), Q_{ef}^g(N)) \end{array}$$

where $\mu_f = \bigoplus_p \mu_{f,p}$, and $\mu_{ef} = \bigoplus_p \mu_{ef,p}$. This permits us to glue the μ_f together over $\mathrm{Proj}(R)$ and we thus obtain the desired morphism $\mu : \boldsymbol{O}_E^+ \to \mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{O}_M^+, \boldsymbol{O}_N^+)$. To check that μ is an isomorphism is easy; actually one can follow the lines of (2.5.13) in [12]. □

NOTE. On the structure sheaf level this implies that for finitely presented M we have that $\mathbf{Hom}_{\tilde{R}}(\tilde{M}, \tilde{N}) = (\mathbf{HOM}_{O_R^+}(O_M^+, O_N^+))_0$.

(2.6) LEMMA. *Let R be positively graded and generated by R_1 over R_0, then for each $P \in \mathrm{Proj}(R)$ we have $Q_P^g(R) = (Q_P^g(R))_0[x, x^{-1}] \cong (Q_P^g(R))_0[T, T^{-1}]$, where x may be found in $Q_P^g(R)_1$ and where T is an indeterminate.*

PROOF. Our assumptions yield that $R_1 \not\subset P$, for otherwise $R_1^n \subset P$ for all n, so $R_+ \subset P$, contradicting the choice of P; it follows that we may find $x \in R_1 - P$, i.e. $\bar{x}$ is invertible in $Q_P^g(R)$, where $\bar{x}$ is the image of x in $Q_P^g(R)$. If $z \in (Q_P^g(R))_n$ for some positive integer n, then $(\bar{x})^{-n}z \in (Q_P^g(R))_0$, i.e. $z \in (Q_P^g(R))_0(\bar{x})^n$. Now, if we had an algebraic relation over $Q_P^g(R)_0$, then the fact that $\deg \bar{x} = 1$ and that $\bar{x}$ is invertible would lead to a contradiction. □

(2.7) LEMMA. *If $\mathbf{M}$ is a locally projective sheaf of $\tilde{R}$-modules of finite type then there exists a graded R-module M of finite presentation such that $\mathbf{M} = \tilde{M}$.*

PROOF. We already know that $\mathbf{M} = \tilde{N}$ for some finitely generated graded R-module N. Consider a free graded R-module F_1 of finite rank and an exact sequence of graded R-modules $0 \to K \to F_1 \to N \to 0$. This sequence leads to an exact sequence $0 \to \tilde{K} \to \tilde{F}_1 \to \tilde{N} = \mathbf{M} \to 0$ in $\tilde{R}$-Mod. Since $\mathbf{M}$ is locally projective of finite type we know that for each $P \in \mathrm{Proj}(R)$ there is an open neighbourhood U such that the induced exact sequence $0 \to \tilde{K} \mid U \to \tilde{F}_1 \mid U \to \mathbf{M} \mid U \to 0$ splits. It follows that $\tilde{K} \mid U$ is a direct summand of a free sheaf of $\tilde{R} \mid U$-modules of finite rank. Consequently, $\tilde{K}$ is locally projective of finite type over $\tilde{R}$. Let $\{f_\lambda ; \lambda \in L\}$ be a set of homogeneous generators for K. For any finite subset H of L, let K_H be the graded R-submodule of K generated by $\{f_\lambda ; \lambda \in H\}$. Clearly $K = \varinjlim_H K_H$ and $\tilde{K} = \varinjlim_H \tilde{K}_H$.

Since $\mathrm{Proj}(R)$ is quasi compact and K has finite type, there exists a finite $H \subset L$ such that $\tilde{K} = \tilde{K}_H$. Consider the following exact diagram in R-gr:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_H & \longrightarrow & F_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow \varphi & & \\ 0 & \longrightarrow & K & \longrightarrow & F_1 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

The map φ derived from $K_H \subset K$ induces a morphism $\tilde{\varphi} : \tilde{M} \to \tilde{N} = \mathbf{M}$. At $P \in \mathrm{Proj}(R)$ the morphism $\tilde{\varphi}_P$ reduces to $Q_P^g(\varphi)_0 : Q_P^g(M)_0 \to Q_P^g(N)_0$. But from $\tilde{K}_H = \tilde{K}$, it follows that $Q_P^g(K_H)_0 = Q_P^g(K)_0$.

Moreover, $(Q_P^g(-))_0 = (-)_{(P)}$ is an exact functor, because it is the composition of exact functors, implying that $(Q_P^g(\varphi))_0$ is an isomorphism. Now $\tilde{\varphi}_P$ is an isomorphism for each $P \in \mathrm{Proj}(R)$, therefore $\tilde{\varphi}$ is an isomorphism. □

(2.8) PROPOSITION. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be locally projective sheaves of finite type over $\tilde{R}$; put $\boldsymbol{M}^+ = \boldsymbol{M} \otimes_{\tilde{R}} \boldsymbol{O}_R^+$ resp. $\boldsymbol{N}^+ = \boldsymbol{N} \otimes_{\tilde{R}} \boldsymbol{O}_R^+$, then*

$$\mathbf{Hom}_{\tilde{R}}(\boldsymbol{M}, \boldsymbol{N}) \otimes_{\tilde{R}} \boldsymbol{O}_R^+ = \mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{M}^+, \boldsymbol{N}^+).$$

PROOF. There is a canonical sheaf morphism

$$\lambda : \mathbf{Hom}_{\tilde{R}}(\boldsymbol{M}, \boldsymbol{N}) \otimes_{\tilde{R}} \boldsymbol{O}_R^+ = \mathbf{HOM}_{\boldsymbol{O}_R^+}(\boldsymbol{M}^+, \boldsymbol{N}^+),$$

hence it will be sufficient to check that for each $P \in \mathrm{Proj}(R)$ the local morphism λ_P at P is an isomorphism. From (2.8) we retain that $\boldsymbol{M} = \tilde{M}$, $\boldsymbol{N} = \tilde{N}$ for M and N finitely presented over R. Now, using the notation $(-)_{(P)}$ for $Q_P^g(-)_0$ in order not to overload notations, we have

$$\begin{aligned}(\mathbf{Hom}_{\tilde{R}}(\boldsymbol{M}, \boldsymbol{N}) \otimes_{\tilde{R}} \boldsymbol{O}_R^+)_P &= (\mathbf{Hom}_{\tilde{R}}(\tilde{M}, \tilde{N}))_P \otimes_{R_P} \boldsymbol{O}_{R,P}^+ \\ &= (\mathrm{HOM}_R(M, N)^{\sim})_P \otimes_{R_{(P)}} Q_P^g(R) \\ &= (\mathrm{HOM}_R(M, N))_{(P)} \otimes_{R_{(P)}} Q_P^g(R).\end{aligned}$$

On the other hand, since $\boldsymbol{M}$ is finitely presented, we have

$$\begin{aligned}(\mathbf{Hom}_{\tilde{R}}(\boldsymbol{M}, \boldsymbol{N}) \otimes_{\tilde{R}} \boldsymbol{O}_R^+)_P &= \mathrm{Hom}_{\tilde{R}_P}(\boldsymbol{M}_P, \boldsymbol{N}_P) \otimes_{R_{(P)}} Q_P^g(R) \\ &= \mathrm{Hom}_{R_{(P)}}(M_{(P)}, N_{(P)}) \otimes_{R_{(P)}} Q_P^g(R)\end{aligned}$$

which is equal to $\mathrm{Hom}_{Q_P^g(R)}(M_{(P)} \otimes_{R_{(P)}} Q_P^g(R), N_{(P)} \otimes_{R_{(P)}} Q_P^g(R))$, since $Q_P^g(M)$ has finite presentation over $Q_P^g(R)$, hence to $\mathrm{Hom}_{Q_P^g(R)}(Q_P^g(M), Q_P^g(N)) = \mathrm{HOM}_{Q_P^g(R)}(Q_P^g(M), Q_P^g(N))$, by (2.4), the last equality holding because $Q_P^g(M)$ is a $Q_P^g(R)$-module of finite type, cf. [12]. But, the same argument yields that $Q_P^g(M) = (\tilde{M} \otimes_{\tilde{R}} \boldsymbol{O}_R^+)_P$ and similarly for $Q_P^g(N)$, which finally yields that

$$\begin{aligned}(\mathbf{Hom}_{\tilde{R}}(\boldsymbol{M}, \boldsymbol{N}) \otimes_{\tilde{R}} \boldsymbol{O}_R^+)_P &= \mathrm{HOM}_{Q_P^g(R)}(\boldsymbol{M}_P^+, \boldsymbol{N}_P^+) \\ &= (\mathbf{HOM}_{\boldsymbol{O}_P^+}(\boldsymbol{M}^+, \boldsymbol{N}^+))_P.\end{aligned}$$

(2.9) NOTE. Choose a finite presentation for M, say $F_2 \xrightarrow{\psi} F_1 \to M \to 0$. Let $\sigma_{R_+}^g$ denote the kernel functor (cf. [11], [37]) associated to the filter of ideals generated by R_+ and let $Q_{R_+}^g$ be the associated graded localization functor (cf. [23]).

Since R_+ is generated by a finite number of elements of degree 1 it follows that σ_{R_+} is of finite type and $Q_{R_+}^g$ commutes with direct sums. It follows easily that

Coker $Q^g_{R_+}(\psi)$ is finitely presented in $Q^g_{R_+}(R)-\mathrm{gr}$, whilst $\tilde{M} = \mathbf{M}$. As usual, if σ is a kernel functor in R-mod or R-gr, then M is said to be *σ-finitely generated* resp. *σ-finitely presented* if M is σ-torsion free and there exists $N \subset M$ which is finitely generated resp. presented and such that M/N is σ-torsion.

(2.10) Corollary. *If $\mathbf{M}$ is locally projective of finite type over $\tilde{R}$, then $\Gamma_*(\mathbf{M})$ is $\sigma^g_{R_+}$-finitely presented.*

Proof. This follows from the foregoing, the fact that $\mathbf{M} = \Gamma_*(\mathbf{M})^\sim$ and the property that $\sigma^g_{R_+} = \inf\{\sigma^g_P; P \in \mathrm{Proj}(R)\}$. □

(2.11) Proposition. *For any graded R-module M we have $Q^g_{R_+}(M) = \Gamma_*(\tilde{M})$.*

Proof. Consider the following commutative diagram of graded R-modules

$$\begin{array}{ccc} M & \xrightarrow{\quad j \quad} & Q^g_{R_+}(M) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \Gamma_*(\tilde{M}) & \xrightarrow[\Gamma_*(\tilde{j})]{} & \Gamma_*(Q^g_{R_+}(M)^\sim) \end{array}$$

It is easily verified that $\Gamma_*(\tilde{j})$ is completely determined by j, i.e. $\Gamma_*(\tilde{j}) = \varphi$ is the unique map making the above diagram into a commutative one. Put $N = Q^g_{R_+}(M)$, then note first that $\mathcal{O}^+_M = \mathcal{O}^+_N$. Indeed the localizing morphism j induces a morphism $\mathcal{O}^+_j : \mathcal{O}^+_M \to \mathcal{O}^+_N$ in the usual way. For $P \in \mathrm{Proj}(R)$, the local morphism $\mathcal{O}^+_{j,P} : \mathcal{O}^+_{M,P} \to \mathcal{O}^+_{N,P}$ reduces to $Q^g_P(j) : Q^g_P(M) \to Q^g_P(N)$. Since $R_+ \not\subset P$, it follows that $\sigma^g_{R_+} \leqq \sigma^g_P$, hence $Q^g_P(N) = Q^g_P(M)$ and $Q^g_P(j)$ is the identity. Consequently $\mathcal{O}^+_j$ is the identity too. From $\mathcal{O}^+_M = \mathcal{O}^+_N$ it follows that $\tilde{M} = \tilde{N}$, hence $\Gamma_*(\tilde{j})$ is actually that identity on $\Gamma_*(\tilde{M})$. For any $K \in R$-gr, the global sections of $\mathcal{O}^+_K$ are given by $\varinjlim \mathrm{HOM}_R(R^n_1, K) \xrightarrow{\sim} Q^g_{R_+}(K)$. Since it is also clear that $\mathcal{O}^+_M = \mathcal{O}^+_{\Gamma_*(\tilde{M})}$, we thus obtain a canonical morphism

$$\gamma : \Gamma_*(\tilde{M}) \to \Gamma(\mathrm{Proj}(R), \mathcal{O}^+_{\Gamma_*(\tilde{M})}) = \Gamma(\mathrm{Proj}(R), \mathcal{O}^+_M) = Q^g_{R_+}(M).$$

Consider the following two commutative diagrams in R-gr:

$$\begin{array}{ccc} M & \xrightarrow[\alpha]{} & \Gamma_*(\tilde{M}) \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle\gamma} & \\ Q^g_{R_+}(M) & & \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\ j\ } & Q^g_{R_+}(M) \\ {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle\beta} & \\ \Gamma_*(\tilde{M}) & & \end{array}$$

Now $(\beta\gamma)\alpha = \beta(\gamma\alpha) = \beta j = \alpha$, hence $\beta\gamma = 1_{\Gamma_*(\tilde{M})}$, because there is only one extension $\Gamma_*(\tilde{j})$ of j. Next, $(\gamma\beta)j = \gamma(\beta j) = \gamma\alpha = j$, hence $\gamma\beta = 1_N$ since 1_N is the unique morphism $N \to N$ extending the identity on $j(M)$. It follows that $\Gamma_*(\tilde{M}) = Q^g_{R_+}(M)$. □

(2.12) COROLLARY. *Let A be any commutative ring and $X = \mathbf{P}^n_A = \mathrm{Proj}(A[X_0, \cdots, X_n])$, then $\Gamma_*(O_X) = A[X_0, \cdots, X_n]$.*

PROOF. One easily sees that for $R = A[X_0, \cdots, X_n]$ we have $Q^g_{R_+}(R) = R$. □

The following series of lemmas aims to give a characterization of locally projective sheaves of $\tilde{R}$-modules of finite type.

(2.13) LEMMA. *Let R be a graded ring such that $RR_1 = R$; if M is a graded R-module with the property that M_0 is a finitely generated R_0-module, then M is a finitely generated projective R-module.*

PROOF. This follows immediately from (2.4). □

(2.14) COROLLARY (2.14.1). *Let R be positively graded and generated by R_1 over R_0; if $M \in R$-gr and $P \in \mathrm{Proj}(R)$ are such that $M_{(P)}$ is a finitely generated projective $R_{(P)}$-module, then $Q^g_P(M)$ is finitely generated and projective over $Q^g_P(R)$.*

(2.14.2) *If $\tilde{M}$ is locally projective of finite type over $\tilde{R}$, then for each $P \in \mathrm{Proj}(R)$, we have that $Q^g_P(M)$ is a finitely generated projective $Q^g_P(R)$-module.* □

(2.15) LEMMA. *Let R be any positively graded ring and let $u : M \to N$ be in $\mathrm{HOM}_R(M, N)$ for some graded R-modules M and N. Let $P \in \mathrm{Proj}(R)$, then*

(2.15.1) *if N is finitely generated and $Q^g_P(u): Q^g_P(M) \to Q^g_P(N)$ is surjective then we can find $f \in h(R)$ such that $P \in X_+(f)$ and $Q^g_f(u): Q^g_f(M) \to Q^g_f(N)$ is surjective;*

(2.15.2) *if M is finitely generated and N is finitely presented and if $Q^g_P(u): Q^g_P(M) \to Q^g_P(N)$ is bijective, then we may find $f \in h(R)$ with $P \in X_+(f)$ and such that $Q^g_f(u): Q^g_f(M) \to Q^g_f(N)$ is bijective;*

(2.15.3) *if N is finitely presented and $Q^g_P(R)$ then there exists $f \in h(R - P)$ such that $Q^g_f N)$ is free over $Q^+_f(R)$.* □

(2.16) Let us call a graded R-module M a $\sigma^g_{R_+}$-quasi projective module if $Q^g_P(M)$ is a projective $Q^g_P(R)$-module for all $P \in \mathrm{Proj}(R)$. Using (2.16) and inspiring oneself on the affine case, one would be tempted to expect $\sigma^g_{R_+}$-quasiprojectivity to imply $\sigma^g_{R_+}$ projectivity in the following sense. Let σ be an arbitrary kernel functor and N an R-module (you may take everything to be graded, if you wish), then N is said to be σ-projective if for each surjective map

$\pi : M_1 \to M_2$ between σ-torsion free R-modules and each map $f : N \to M_2$ we may find $N_1 \subset N$ with N/N_1 being σ-torsion and $f_1 : N_1 \to M_1$ making the following diagram commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & N_1 & \longrightarrow & N & & \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \\ & & M & \xrightarrow[\pi]{} & M_2 & \longrightarrow & 0 \end{array}$$

Now, using $\sigma^g_{R_+}$-quasiprojectivity, it is only possible to deduce the following weaker version of $\sigma^g_{R_+}$-projectivity: if $\pi : M_1 \to M_2$ is a surjective morphism between σ-torsion free R-modules and if for an R-module M we let $\mathrm{Hom}_R(M, \pi) : \mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_2)$ denote the canonical morphism, then for each $f \in \mathrm{Hom}_R(M, M_2)$ we may find a positive integer n such that $R_+^n f \subset \mathrm{Im}\,\mathrm{Hom}_R(M, \pi)$. Nevertheless, we have the following

(2.17) PROPOSITION. *Let M be a sheaf $\tilde{R}$-module, where R is a positively graded ring generated by R_1 over R_0, then the following statements are equivalent*:

(2.17.1) $\Gamma_*(M)$ *is a graded $\sigma^g_{R_+}$-finitely presented $\sigma^g_{R_+}$-quasiprojective $Q^g_{R_+}(R)$-module*;

(2.17.2) M *is locally projective of finite type.*

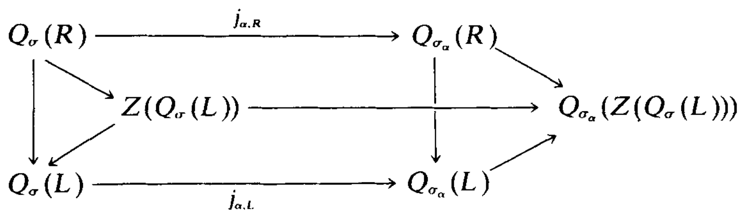
PROOF. (1) $\Rightarrow$ (2). Pick a graded finitely presented $M \subset \Gamma_*(M)$ such that $\Gamma_*(M)/M$ is $\sigma^g_{R_+}$-torsion. Obviously $\tilde{M}$ is finitely presented over $(Q^g_{R_+}(R))^{\sim} = \tilde{R}$ and $M = \tilde{M}$ since $Q^g_P(\Gamma_*(M)/M) = 0$ for all $P \in \mathrm{Proj}(R)$. Moreover, $Q^g_P(\Gamma_*(M))$ is projective over $Q^g_P(R)$, hence $M_P = Q^g_P(\Gamma_*(M))_0 = \Gamma_*(M)_{(P)}$ is $\tilde{R}_P = R_{(P)}$-projective by the foregoing. Thus by (1.2) it follows that M is locally projective of finite type.

(2) $\Rightarrow$ (1). This is an easy consequence of (2.10) and (2.14). □

Let us now look at locally separable sheaves of algebras over $\mathrm{Proj}(R)$. We will start with the following lemma, which we will prove in full generality for later use.

(2.18) PROPOSITION. *Let L be any R-algebra with R contained in the center of L. Consider a family of kernel functors $\{\sigma_\alpha ; \alpha \in A\}$ in R-mod, such that each σ_α has the property that $Q_{\sigma_\alpha}(L)$ is a ring with $Z(Q_{\sigma_\alpha}(R))$, then if $\sigma = \inf\{\sigma_\alpha ; \alpha \in A\}$ we have that $Q_\sigma(L)$ is a ring with center $Q_\sigma(R)$.*

PROOF. Although the result holds for any ring, we will only prove this in case R is a domain. One easily checks that $Q_\sigma(L)$ is a ring and that $Q_\sigma(Z(L)) \subset Z(Q_\sigma(L))$. Consider the following diagram of ring homomorphisms



Here, $R\subset Z(L)$ yields $Q_\sigma(R)\subset Q_\sigma(Z(L))\subset Z(Q_\sigma(L))$ and $j_{\alpha,L}$ maps $Z(Q_\alpha(K))$ into $Z(Q_{\sigma_\alpha}(L))=Q_{\sigma_\alpha}(R)$. Moreover $Q_{\sigma_\alpha}(Z(Q_\sigma(L)))\subset Z(Q_{\sigma_\alpha}(Q_\sigma(L)))=Z(Q_{\sigma_\alpha}(L))=Q_{\sigma_\alpha}(R)$. For each α we obtain $Q_{\sigma_\alpha}(Z(Q_\sigma))=Q_{\sigma_\alpha}(Q_\sigma(R))$. Consequently, $Z(Q_\sigma(L))/Q_\sigma(R)$ is a σ_α-torsion R-module for all $\alpha\in A$ hence also a σ-torsion R-module. But then $Z(Q_\sigma(L))\subset Q_\sigma(Z(Q_\sigma(L)))=Q_\sigma(Q_\sigma(R))=Q_\sigma(R)$, i.e. $Z(Q_\sigma(L))=Q_\sigma(R)$. □

(2.19) Corollary (of the proof). *The analogous graded statement is valid too.* □

Assume again that R is a positively graded domain generated by a finite number of elements of degree 1 over R_0.

(2.20) Lemma. *Let S be a graded R-algebra with the property that $SS_1=S$ and consider a graded S-algebra A; if $Z(A_0)=S_0$, then $Z(A)=S$.*

Proof. If $z\in Z(A_0)$ and $a\in A_n$, put $b=az$, then $Sb=SS_{-n}b=0$, therefore $Z(A_0)=Z(A)_0$. If $y\in Z(A)_m$, then $Sy=SS_{-m}y\subset SZ(A)_0$, hence $Sy\subset SS_0=S$ and we find that indeed $Z(A)=S$. □

(2.21) Corollary. *If for some graded R-algebra L we know that $\tilde{L}$ is a central sheaf of $\tilde{R}$-algebras, then $\mathbf{O}_L$ is a central sheaf of $\mathbf{O}_R$-algebras.*

(2.22) Lemma. *Let S be a graded R-algebra which is flat as an S_0-module and which has the property that $SS_1=S$. A graded S-algebra A which has the property that A_0 is an Azumaya algebra over S_0 is a graded Azumaya algebra over S.*

Proof. We may apply (2.4) and (2.13) to derive that A is S-algebra isomorphic to $A_0\otimes_{S_0}S$. It follows that A is a graded Azumaya algebra over S.

□

(2.23) Corollary. *If for some graded R-algebra L we have that L is a locally separable sheaf of $\tilde{R}$-algebras, then for each $P\in\mathrm{Proj}(R)$ the algebra $Q_P^g(L)$ is central separable over $Q_P^g(R)$.* □

(2.24) From (2.19) it follows from each finitely presented central sheaf of

$\tilde{R}$-algebras $\boldsymbol{L}$ over $\mathrm{Proj}(R)$ that $\Gamma_*(\boldsymbol{L})$ is a graded central $\Gamma_*(\tilde{R})$-algebra. By our assumptions on R it is clear that R is noetherian if and only if R_0 is noetherian. Let us assume this from here on, then any sheaf of $\tilde{R}$-modules $\boldsymbol{M}$ which is of finite type is of the form $\tilde{M}$ for some graded R-module M which is finitely presented and σ_{R_+}-torsion free. Indeed, $\boldsymbol{M} = \tilde{N}$ for some $N \in R$-gr which is of finite type, choose $M = N/\sigma^g_{R_+}N$, then M is finitely generated, hence finitely presented and obviously M is σ_{R_+}-torsion free. It is clear that $\boldsymbol{M} = \tilde{M}$.

(2.25) Lemma. *Let L be a graded $\sigma^g_{R_+}$-quasiprojective, $\sigma^g_{R_+}$-torsion free R-module of finite presentation and let $\boldsymbol{L} = \tilde{L}$, then*

$$\mathrm{Hom}_{\Gamma_*(\tilde{R})}(\Gamma_*(\boldsymbol{L}), \Gamma_*(\boldsymbol{L})) = \Gamma_*(\mathrm{Hom}_{\tilde{R}}(\boldsymbol{L}, \boldsymbol{L})).$$

Proof. It is clear that L is $\sigma^g_{R_+}$-flat in the sense of [42]. Indeed, since R is commutative we only have to verify that if $i : M' \to M$ is a monomorphism of graded R-modules, then $\mathrm{Ker}(L \otimes_R i)$ is $\sigma^g_{R_+}$-torsion, but this follows immediately from the fact that L is $\sigma^g_{R_+}$-quasiprojective, $\sigma^g_{R_+} = \inf\{\sigma^g_P; P \in \mathrm{Proj}(R)\}$, and the fact that localization at multiplicative systems commutes with tensor-products. From loc. cit. it then follows that $\mathrm{HOM}_{Q^g_{R_+}(R)}(Q^g_{R_+}(L), Q^g_{R_+}(L)) = \mathrm{HOM}_R(Q^g_{R_+}(L), Q^g_{R_+}(L))$ is $\sigma^g_{R_+}$-closed. Let us now first show that $\mathrm{HOM}_{Q^g_{R_+}}(Q^g_{R_+}(L), Q^g_{R_+}(L)) = Q^g_{R_+}(\mathrm{HOM}_R(L, L))$.

From $\mathrm{HOM}_R(Q^g_{R_+}(L), Q^g_{R_+}(L)) = \mathrm{HOM}_R(L, Q^g_{R_+}(L))$ it follows that this assertion will be established if we are able to prove that in the exact sequence

$$0 \longrightarrow K \longrightarrow \mathrm{HOM}_R(L, L) \longrightarrow \mathrm{HOM}_R(L, Q^g_{R_+}(L)) \longrightarrow T \longrightarrow 0$$

both K and T are $\sigma^g_{R_+}$-torsion. First suppose that $\varphi : L \to L$ is in $\mathrm{Ker}\,\Delta = K$. Let us write $j : L \to Q^g_{R_+}(L)$ for the canonical localization morphism, then $\varphi \in K$ yields $j\varphi = 0$; i.e. $\varphi(L) \subset \mathrm{Ker}\, j = \sigma^g_{R_+}L$. Since L is finitely generated we may find a positive integer $n \in \mathbf{N}$ such that $R^n_+\varphi(L) = 0$, i.e. $R^n_+\varphi = 0$ and $\varphi \in \sigma^g_{R_+}\mathrm{HOM}_R(L, L)$, proving that K is $\sigma^g_{R_+}$-torsion. Next, in order to prove that T is $\sigma^g_{R_+}$-torsion, it suffices to verify that for any $\varphi : L \to Q^g_{R_+}(L)$ there is an $L_1 \subset L$ with L/L_1 being $\sigma^g_{R_+}$-torsion and a morphism $\varphi_1 : L_1 \to L$ such that the following diagram is commutative;

$$\begin{array}{ccc} L_1 & \longrightarrow & L \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi} \\ L & \xrightarrow[j]{} & Q^g_{R_+}(L) \end{array}$$

Now, if $L' = \varphi(L) \subset Q^g_{R_+}(L)$, then, as L' is finitely generated, we may find a positive integer $n \in \mathbf{N}$ such that $R_+^n L' \subset L$. Put $L_1 = R_+^n L'$ and $\varphi_1 = \varphi \mid L_1$, then (L_1, φ_1) satisfies our requirements.

To finish the proof, observe that

$$\begin{aligned}\mathrm{Hom}_{\Gamma_*(\tilde{R})}(\Gamma_*(\boldsymbol{L}), \Gamma_*(\boldsymbol{L})) &= \mathrm{HOM}_{Q^g_{R_+}(R)}(Q^g_{R_+}(L), Q^g_{R_+}(L)) \\ &= Q^g_{R_+}(\mathrm{HOM}_R(L, L)) = \Gamma_*((\mathrm{HOM}_R(L, L))^{\sim}) \\ &= \Gamma_*(\mathbf{Hom}_{\tilde{R}}(\boldsymbol{L}, \boldsymbol{L})).\end{aligned}$$

The last equality holds because of (2.3.3). □

(2.26) Corollary. *If $\boldsymbol{L}$ is a sheaf of R-modules which is locally finite then* $\mathrm{Hom}_{\Gamma_*(\tilde{R})}(\Gamma_*(\boldsymbol{L}), \Gamma_*(\boldsymbol{L})) = \Gamma_*(\mathrm{Hom}_{\tilde{R}}(\boldsymbol{L}, \boldsymbol{L}))$. □

(2.27) Lemma. *Let $\boldsymbol{L}$ and $\boldsymbol{L}'$ be sheaves of $\tilde{R}$-modules, then*

$$\Gamma_*(\boldsymbol{L} \otimes_{\tilde{R}} \boldsymbol{L}) = Q^g_{R_+}(\Gamma_*(\boldsymbol{L}) \otimes_{\Gamma_*(\tilde{R})} \Gamma_*(\boldsymbol{L})) = \Gamma_*((\Gamma_*(\boldsymbol{L}) \otimes_{\Gamma_*(\tilde{R})} \Gamma_*(\boldsymbol{L}))^{\sim}). \quad \square$$

Let us call a graded R-algebra A a *$\sigma^g_{R_+}$-quasi Azumaya algebra* if it is $\sigma^g_{R_+}$-closed, $\sigma^g_{R_+}$-quasi projective and $\sigma^g_{R_+}$-finitely generated, if $Z(A) = Q^g_{R_+}(R)$ and if the canonical map $A^e = A \otimes_R A^0 \to \mathrm{HOM}_R(A, A)$ induces an isomorphism $Q^g_{R_+}(A^e) \xrightarrow{\sim} \mathrm{HOM}_R(A, A)$. Then we have:

(2.28) Proposition. *There is a bijective correspondence between locally separable sheaves of $\tilde{R}$-algebras on* Proj(R) *and graded $\sigma^g_{R_+}$-quasi Azumaya algebra over R.* □

Using these $\sigma^g_{R_+}$-Azumaya algebras, it is now possible to introduce a notion of "relative" graded Brauer group. However, as this would lead us too far, we will not go into the details here. Moreover, in the next section we will show that in the case of a normal curve this relative Brauer group is just the usual Brauer group B^g of the homogeneous coordinate ring of the curve under consideration, cf. [41].

3. Normal varieties

(3.1) Let us now turn to the geometrical situation: we will consider normal schemes. Recall that a scheme X is *normal* if its local rings are integrally closed domains. If X is affine then it is normal exactly when its affine coordinate ring is integrally closed. If X is one dimensional, then X is normal exactly when it is nonsingular. Let C be a commuative ring and X a closed subscheme of $\mathbf{P}^r_C$, let $S[X]$ be the ring $C[X_0, \cdots, X_r]/I$ where $I = \Gamma_*(\boldsymbol{J}_X)$, where $\boldsymbol{J}_X$ is the sheaf of ideals defining X. We call X *projectively normal* (for the given embedding) if $S[X]$ is integrally closed. For these notions and the following result we refer to Hartshorne [15].

(3.2) PROPOSITION. *Let k be a field and A a finitely generated k-algebra; if X is a connected, normal subscheme of $\mathbf{P}^r_A$, then*

(3.2.1) $S' = \bigoplus_{n \geqq 0} \Gamma(X, O_X(n))$ *is the integral closure of the domain* $S[X]$;

(3.2.2) *for $d \in \mathbf{N}$ large enough, we have* $S[X]_d = S'_d$;

(3.2.3) *an arbitrary closed subscheme $X \subset \mathbf{P}^r_A$ is projectively normal if and only if X is normal and for every $n \in \mathbf{N}$ the natural map $\Gamma(\mathbf{P}^r_k, O_{\mathbf{P}^r_k}(n)) \to \Gamma(X, O_X(n))$ is surjective.* □

A closed subscheme Y of $\mathbf{P}^n_k$ is a *complete intersection* if the homogeneous ideal I of Y in $S = k[X_0, \cdots, X_n]$ may be generated by $r = \operatorname{codim}(Y, \mathbf{P}^r_k)$ elements. It is straightforward to establish that any normal complete intersection of dimension $\geqq 1$ in $\mathbf{P}^n_k$ is also projectively normal.

(3.3) LEMMA. *Let R be generated by a finite number of elements of degree 1 over R_0; if R is a domain then every homogeneous element of non-negative degree in $Q^g_{R_+}(R)$ is integral over R.*

PROOF. Let $\{f_0, \cdots, f_n\} \subset R_1$ generate R as an R_0-algebra. Pick $f \in Q^g_{R_+}(R)$ homogeneous of degree $d \geqq 0$, then for some $n \in \mathbf{N}$ we have $R^n_+ f \subset R$. In particular $R^n_1 f \subset R_{n+d}$; but our assumptions imply $R_{n+d} = R_1^{n+d}$, so we obtain $R^n_1 f^2 \subset R_1^{n+d} f \subset R_1^{n+2d}$. Consequently, $R^n_1 f^s \subset R$ for all $s \in \mathbf{N}$. In particular $f^n_0 f^s \subset R$, hence $R[f] \subset f_0^{-n} R$. Since $f_0^{-n} R$ is a noetherian R-module, it follows that $R[f]$ is finitely generated, hence f is integral over R. □

(3.4) COROLLARY. *Under the foregoing assumptions, if R_0 is a field, then so is* $(Q^g_{R_+}(R))_0 = \Gamma(\operatorname{Proj}(R), \tilde{R})$.

PROOF. Pick $0 \neq x \in (Q^g_{R_+}(R))_0$; then a is integral over the field R_0 and contained in a domain, i.e. x is invertible. □

(3.5) COROLLARY. *Let R be an affine graded domain over a field k, then $Q^g_{R_+}(R)$ is positively graded in* $\operatorname{Proj}(R) \neq \{0\}$.

PROOF. By assumption R is of the form $k[X_0, \cdots, X_n]/I$ where I is a graded prime ideal of $k[X_0, \cdots, X_n]$. Suppose we may find $0 \neq r \in Q^g_{R_+}(R)_{-n}$ with $n > 0$; since $R_n \neq 0$ we may pick a nonzero f in R_n and so $fr \in Q^g_{R_+}(R)_0$ is invertible in $Q^g_{R_+}(R)_0$ by (3.4), hence r is invertible in $Q^g_{R_+}(R)$. For arbitrary $r' \in Q^g_{R_+}(R)_m$ we may multiply r' by r^q for some q large enough such that

$r^q r' \in Q^g_{R_+}(R)_{-p}$ for some $p > 0$. Since $r^q r' \neq 0$ the foregoing argument yields that $r^q r'$ is invertible in $Q^g_{R_+}(R)$ and therefore r' is invertible in $Q^g_{R_+}(R)$. Consequently $Q^g_{R_+}(R) = Q^g_P(R)$ for all $P \in \mathrm{Proj}(R)$, meaning that $\mathrm{Proj}(R) = \{0\}$. This situation having been excluded, this proves the assertion. □

(3.6) COROLLARY. *If R is the homogeneous coordinate ring of a connected normal projective k-variety, then $Q^g_{R_+}(R)$ is the integral closure of R.*

PROOF. Apply (3.5), (3.2) and (2.11). □

(3.7) COROLLARY. *Let R be an affine graded domain over a field k, then $Q^g_{R_+}(R)$ is a graded noetherian domain.*

PROOF. We know that $Q^g_{R_+}(R)$ is positively graded and that for large degrees $d \geqq d_0$ we have $(Q^g_{R_+}(R))_d = R_d$, cf. (2.11) and (2.3.7). By Serre's theorem, cf. [15, 25], however, we know that the part of $Q^g_{R_+}(R)$ of degree lower than d_0 is a finitely generated R_0-module, hence noetherian as well. □

NOTE. Under the above assumptions it does not necessarily follow that $Q^g_{R_+}(R)$ is affine over $Q^g_{R_+}(R)_0$ (in the sense that elements of degree 1 generate it); however, we know that for d large enough we have $Q^g_{R_+}(R)_d = R_1^d$. This will be seen to be amply sufficient for our constructions to be possible.

(3.8) LEMMA. *If A is a $\sigma^g_{R_+}$-quasi Azumaya algebra, then for all $P \in \mathrm{Proj}(Q^g_{R_+}(R))$ we have that $Q^g_P(A)$ is a graded Azumaya algebra over $Q^g_P(Q^g_{R_+}(R))$.*

PROOF. Since $Q^g_{R_+}(R)$ is a graded integral over R it follows that $p = P \cap R$ is in $\mathrm{Proj}(R)$ if $P \in \mathrm{Proj}(Q^g_{R_+}(R))$. Indeed, it suffices to verify that if a graded prime ideal P of $Q^g_{R_+}(R) = R'$ contains R_+ then P also contains R'_+. Now, any $x \in h(R'_+)$ satisfies a relation of the form $x^n = r_1 x^{n-1} + \cdots + r_n$ for some $r_i \in h(R_+)$. So, if P contains R_+, then P contains x^n, hence $x \in P$. It follows that P contains $h(R'_+)$ hence R'_+! Now,

$$Q^g_P(A) = Q^g_P(Q^g_{R_+}(R)) \otimes_{Q^g_P(R)} Q^g_P(A).$$

We know that $Q^g_p(A)$ is a graded Azumaya algebra over Q^g_p, hence it follows that $Q^g_P(A)$ is a graded $Q^g_P(Q^g_{R_+}(R))$-Azumaya algebra, as asserted. □

(3.9) REMARK. The minimal prime ideals of R which are graded are all contained in $\mathrm{Proj}(R)$ — this holds for any positively graded ring R such that R_0 is a domain. Consequently, the assumption that $Q^g_P(A)$ be an Azumaya algebra over $Q^g_P(R)$ for all minimal and graded prime ideals of R will be fulfilled in our context.

If $M \in R$-mod, then $M^* = \mathrm{Hom}_R(M, R)$ is the *dual* of M and M^{**} is the double dual. We say that M is *reflexive* if $M \cong M^{**}$.

(3.10) LEMMA. *If S is a noetherian domain of global dimension at most 2 then every finitely generated reflexive S-module is projective.*

PROOF. Cf. [4]. □

(3.11) LEMMA. *Suppose that R is as before but also integrally closed and let A be a graded algebra such that $Q^g_{R_+}(A) = A$. If for every $P \in \mathrm{Proj}(R)$ we have that $Q^g_P(A)$ is a graded Azumaya algebra over $Q^g_P(R)$, then A is an R-order in its total ring of fractions $Q(A)$.*

PROOF. For each $P \in \mathrm{Proj}(R)$ the localizing morphism $j_P : A \to Q^g_P(A)$ is a central extension with kernel $\sigma^g_P(A)$. Since $Q^g_P(A)$ is an Azumaya algebra over the domain $Q^g_P(R)$ it is in particular a prime ring, therefore $\sigma^g_P(A)$ is a prime ideal of A. Since $\sigma^g_{R_+}(A) = 0 = \bigcap \sigma^g_P(A)$ it follows that A is a semiprime ring. Now, let I, J be any nonzero ideals of A such that $IJ = 0$, then $IJ = JI = 0$ because A is semiprime.

If $j_P(I)$ were zero for all $P \in \mathrm{Proj}(R)$, then $I \subset \sigma^g_{R_+}(A) = 0$ and similarly for J. Fix P such that $j_P(I)Q^g_P(A) \neq 0$. Since $Q^g_P(A)$ is a prime P.I. ring it follows that the nonzero ideal $j_P(I)Q^g_P(A)$ intersects the center $Q^g_P(R)$ nontrivially. Pick $0 \neq z_P \in Q^g_P(R) \cap j_P(I)Q^g_P(A)$, then $0 \neq H_P z_P \subset j_P(R) \cap j_P(I)$ for some $H_P \not\subset P$ in R. Pick some $z'_P = h_P z_P \neq 0$ in $H_P z_P$ and let $z \in I$ be such that $j_P(z) = z'_P$, then, from $j_P(z) \in j_P(R)$, it follows that $z = r + \lambda$ for some $\lambda \in \sigma_P(A)$. Now $H'_P \lambda = 0$ for some $H'_P \not\subset P$ yields that $H'_P z = H'_P r \subset R$. Since R is a domain, we have that $H'_P r \neq 0$ and hence $H'_P z \subset I$ is in $I \cap R$. Consequently, nonzero ideals of A intersect the center nontrivially. However, it then follows that $IJ = 0$ implies $(I \cap R)(J \cap R) = 0$, i.e., either I or J is zero and this proves that A is a prime ring and all morphisms j_P are injective. Therefore A is a prime P.I. algebra with center $Z(A) = R$, by (2.19). From Posner's theorem (cf. [23]) it follows that $Q(A) = A \otimes_R Q(R) = A \otimes_R K$ is a finite dimensional simple algebra with center K. We thus have the following diagram of inclusions:

$$\begin{array}{ccc} K & \longrightarrow & Q(A) \\ \downarrow & & \downarrow \\ Q^g_P(R) & \longrightarrow & Q^g_P(A) \end{array}$$

and each $Q_P^g(A)$ is a $Q_P^g(R)$-order in the K central simple algebra $Q(A)$. Take $z \in A$, then since $z \in Q_P^g(A)$, the minimal polynomial of z in $Q(A)$ over K has coefficients in $Q_P^g(R)$. The latter holds for all $P \in \mathrm{Proj}(R)$, hence the considered minimum polynomial has coefficients in $\bigcap \{Q_P^g(R): P \in \mathrm{Proj}(R)\} = Q_{R_+}^g(R)$. Since R is integrally closed here, $Q_{R_+}^g(R) = R$, i.e., we have $AK = Q_{R_+}^g(A)$ and A is integral over R and thus A is an R-order. □

(3.12) COROLLARY. (3.12.1) *In the above situation A is finitely generated as an R-module, since any order over an integrally closed noetherian domain R in a central simple algebra has to be finitely generated over R*, cf. [24] VI 5.

(3.12.2) *Since $Q_P^g(R)$ is integrally closed in K and $Q_P^g(A)$ is an Azumaya order over $Q_P^g(R)$ in $Q(A)$, it follows that each $Q_P^g(A)$ is a maximal $Q_P^g(R)$-order in $Q(A)$.*

(3.13) Now, by (3.12.1) it follows that A is finitely generated over R so we obtain that $A^* = \mathrm{HOM}_R(A, R)$. Therefore A^* is a graded and finitely generated R-module. Both A^* and A^{**} being graded, it is not necessary to specify whether the dual has been taken in the graded or non-graded sense. Moreover, the "evaluation map" $A \to A^{**} : a \to \hat{a}$, where $\hat{a}(f) = f(a)$ for any $f \in A^*$, is a graded morphism of degree zero. These remarks allow us to prove

(3.14) LEMMA. *Let A and R be as in* (3.16), *then* $A \subset A^{**} \subset Q^g(A)$.

PROOF. Suppose $z = ac^{-1}$ with $a \in Z(A) = R$ is in $h(A^{**})$, where A^{**} is embedded in $Q(A)$ by the classical, ungraded results. By the foregoing remarks, $A \to A^{**}$ is graded of degree zero, so the gradation of A^{**} induces the original one on A. Write $a = a_{i_1} + \cdots + a_{i_n}$ with $i_1 > \cdots > i_n$ and $\deg(a_{i_\mu}) = i_\mu$ and similarly $c = c_{j_1} + \cdots + c_{j_m}$, with $j_1 > \cdots > j_m$ and $\deg(c_{j_\lambda}) = j$. Considering $(c_{j_1} + \cdots + c_{j_m})z = a_{i_1} + \cdots + a_{i_n}$ in A^{**} we obtain $c_{j_\lambda} = a_{i_\mu}$. Hence $z \in Q^g(A)$, so it follows that $A^{**} \subset Q^g(A) \subset Q^g(A)$ since $h(A^{**})$ generates A^{**}. □

For each ideal I of R we denote the largest graded ideal contained in I by I_g.

(3.15) LEMMA. *Let* $\mathbf{P}$ *be the set of minimal prime ideals of R, let* $\mathbf{P}_g = \{P_g ; P \in \mathbf{P}\}$; *if R is noetherian, then*

(3.15.1) $\mathbf{P}_g$ *consists of prime ideals which are minimal and graded*;

(3.15.2) $\mathbf{P}_g \neq \{0\}$ *unless R is a graded field.*

PROOF. (1) Since $0 \subset P_g \subset P$ and P_g is prime it follows that $P \in \mathbf{P}$ implies $P_g = 0$ or $P_g = P$.

(2) If R is not a graded field, then there exists a nonzero homogeneous $u \in h(R)$ which is not invertible in R. Krull's principal ideal theorem yields that

any prime ideal P minimal over Ru is a minimal prime ideal. From $0 \neq Ru \subset P_g$ it then follows that $P = P_g$. □

(3.16) LEMMA. *If R is integrally closed and $M \in R$-mod finitely generated and torsion-free, then $M^{**} = \bigcap_{P \in \mathbf{P}} M_P$.*

PROOF. Cf. [4]. □

At this point we gladly credit J. Van Geel for suggesting the use of the above lemma in the proof of the following proposition.

(3.17) PROPOSITION. *Let A and R be as before, then A is reflexive.*

PROOF. Since A is an R-algebra it is certainly torsion-free. We already know A to be finitely generated, so we may apply (3.16), i.e. $A^{**} = \bigcap_{P \in \mathbf{P}} A_P$. Since A^{**} is graded, we find that $A^{**} \subset \bigcap_{P \in \mathbf{P}_g} Q_P^g(A) = B$. However, if $z \in h(B)$, then we may find a graded ideal $0 \neq I \not\subset P$ for all $P \in \mathbf{P}_g$, such that $Iz \subset A^{**}$. Now, if $P' \in \mathbf{P} - \mathbf{P}_g$ then P' cannot contain the graded ideal I, for otherwise $I \subset P'_g \subset P'$. Since $z \in Q[A]$ is thus such that $Iz \subset A$ for $I \not\subset P$, it follows that $z \in Q_{P'}(A)$ and the latter for $P' \in \mathbf{P} - \mathbf{P}_g$. So $A^{**} \subset B \subset \bigcap_{P \in \mathbf{P}} Q_P(A) = A^{**}$, because B is R-generated by $h(B)$! Let us now establish that $Z(A^{**}) = Z(A) = R$. First if $x \in Z(A)$, then $x \in \bigcap_{P \in \mathbf{P}_g} Z(Q_P^g(A)) \subset Z(A^{**})$. Conversely, if $y \in Z(A^{**})$, then y commutes with every element of A in $Q(A)$. Therefore, $y \in \bigcap_{P \in \mathbf{P}_g} Z(Q_P^g(A)) = \bigcap_{P \in \mathbf{P}_g} Q_P^g(R) = R$, the latter because R is known to be reflexive.

For any $P \in \mathrm{Proj}(R)$, we have that $Q_P^g(A) \subset Q_P^g(A^{**})$ and the fact that A^{**} is an R-order implies that $Q_P^g(A^{**})$ is a $Q_P^g(R)$-order. The maximality of $Q_P^g(A)$ yields $Q_P^g(A) = Q_P^g(A^{**})$ for all $P \in \mathrm{Proj}(R)$. Finally, $A^{**} \subset Q_{R_+}^g(A^{**}) = Q_{R_+}^g(A) = A$ implies $A = A^{**}$. □

NOTE. This may also be proved directly using (2.3) in [44]. The proof given here involves less abstract nonsense.

(3.18) Recall from [41] that a graded R-algebra A is said to be a pseudo-Azumaya algebra if it is reflexive and if the canonical map

$$(A \otimes_R A^{\mathrm{opp}})^{**} \to \mathrm{End}_R(A)$$

is an isomorphism. Two pseudo-Azumaya algebras A and B are said to be similar if we may find reflexive R-modules P and Q and an isomorphism of graded R-algebras

$$A \otimes_R \mathrm{END}_R(P) \simeq B \otimes_R \mathrm{END}_R(Q).$$

The set of similarity classes of pseudo-Azumaya algebras may be endowed with a group structure in the obvious way and one thus obtains the so-called relative Brauer group $B^g(R)$. For more details, cf. loc. cit.

We may now prove

(3.19) THEOREM. *Let $X = \mathrm{Proj}(R)$ be a connected regular projective k-variety of dimension at most two, then there is a bijective correspondence between locally separable sheaves of O_X-algebras and pseudo-Azumaya algebras over $\Gamma_*(\tilde{R})$.*

PROOF. First note that for any finitely generated torsion-free graded $Q^g_{R_+}(R)$-module we have $M \subset Q^g_{R_+}(M) \subset M^{**}$, by (3.16) and the proof of (3.17), since $Q^g_{R_+}(R)$ is noetherian integrally closed by (3.6) and (3.7). It follows that if M is reflexive it is certainly $\sigma^g_{R_+}$-closed. On the other hand, it also follows that if $Q^g_{R_+}(M)$ is reflexive then $Q^g_{R_+}(M) = M^{**}$.

Now, let L be a locally separable sheaf of O_X-algebras, then $A = \Gamma_*(L)$ is reflexive by (3.17) and we know that $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}}) = \mathrm{END}_R(A)$. But $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}})$ is reflexive by (3.17), hence $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}}) = (A \otimes_R A^{\mathrm{opp}})^{**}$, proving that A is a pseudo-Azumaya algebra.

Conversely, let us show that any pseudo-Azumaya algebra is a graded $\sigma^g_{R_+}$-quasi Azumaya algebra, then we may apply (2.28) to finish the proof. So, let A be pseudo-Azymaya algebra, then A is reflexive, hence finitely generated and torsion-free, so it is certainly $\sigma^g_{R_+}$-closed. Since $\Gamma_*(\tilde{R}) = Q^g_{R_+}(R)$ is noetherian, we have that $Q^g_P(A)^{**} = Q^g_P(A)$ in $Q^g_P(R)$-gr, hence we may apply (2.4) to obtain that $Q^g_P(A)_0$ is a reflexive $Q^g_P(R)_0$-module. But then $Q^g_P(A)_0$ is a projective $Q^g_P(R)_0$-module and $Q^g_P(A)$ is graded projective over $Q^g_P(R)$, proving that A is $\sigma^g_{R_+}$-quasiprojective. We thus have proved that A and hence $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}})$ is a $\sigma^g_{R_+}$-finitely generated $\sigma^g_{R_+}$-closed $\sigma^g_{R_+}$-quasiprojective R-module, so we may invoke (2.3) in [44] to derive that $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}})$ is reflexive, hence $Q^g_{R_+}(A \otimes_R A^{\mathrm{opp}}) = (A \otimes_R A^{\mathrm{opp}})^{**} \cong \mathrm{END}_R(A)$ and A is indeed a $\sigma^g_{R_+}$-quasi-Azumaya algebra. This finishes the proof. □

Let us now define a morphism

$$B^g(\Gamma_*(R)) \to B(\mathrm{Proj}(R), \tilde{R})$$

$$[L] \mapsto [\tilde{L}],$$

then the first part of the proof of (3.19) shows that this is a well-defined group morphism for $\mathrm{Proj}(R)$ normal. Moreover, using (2.3) in [44] one may easily verify

(3.20) THEOREM. *Let $\mathrm{Proj}(R)$ be a connected regular projective k-variety of dimension at most two, then $B(\mathrm{Proj}(R), /\tilde{R}) = B^g(\Gamma_*(\tilde{R}))$.*

REMARK. The foregoing result gives a geometric interpretation of the relative Brauer group introduced in [41], at least in the regular case. For the non-normal case we refer to a paper in preparation.

REFERENCES

1. S. A. Amitsur, *Simple algebras and cohomology groups of arbitrary fields*, Trans. Amer. Math. Soc. **90** (1959), 73–112.

2. M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, J. Algebra **11** (1969), 532–563.

3. B. Auslander, *The Brauer group of a ringed space*, J. Algebra **4** (1966), 220–273.

4. M. Auslander and O. Goldman, *Maximal orders*, Trans. Amer. Math. Soc. **97** (1960), 1–24.

5. M. Aulander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

6. G. Azumaya, *On maximally central algebras*, Nagoya Math. J. **2** (1951), 119–150.

7. N. Bourbaki, *Elements de Mathématique, Algèbre*, Chap. 8, Hermann, Paris, 1962.

8. L. N. Childs, *On Brauer Groups of Some Local Rings*, Evanston 1975, Proceedings, Lecture Notes in Math. **549**, Springer-Verlag, Berlin, pp. 1–15.

9. F. De Meyer, *The Brauer Group of Affine Curves*, Evanston 1975, Proceedings, Lecture Notes in Math. **549**, Springer-Verlag, Berlin, pp. 16–24.

10. F. De Meyer and E. Ingraham, *Separable Algebras over Commutative Rings*, Lecture Notes in Math. **181**, Springer-Verlag, Berlin, 1970.

11. P. Gabriel, *Des Catégorie Abéliennes*, Bull. Soc. Math. France **90** (1963), 324–448.

12. A. Grothendieck, *Le Group de Brauer I, (II, III), Dix exposés sur la cohomogie des schémas*, North-Holland, Amsterdam, 1968, pp. 46–65.

13. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tohoku Math. J. **2** (1957), 119–221.

14. A. Grotendieck and J. Dieudonné, *Eléments de géometrie algébrique II*, Publ. Math. I.H.E.S., Paris, 1961.

15. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

16. B. Iversen, *Noetherian Graded Modules, I*, Aarhus Univ. preprint no. 29, 1972.

17. M. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math. **389**, Springer-Verlag, Berlin, 1974.

18. J. Lambek and G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, J. Algebra **25** (1973), 364–389.

19. D. Lazard, *Autour de la platitude*, Bull. Soc. Math. France **97** (1969), 81–128.

20. D. Mumford, *Introduction to Algebraic Geometry*, Preliminary version of first 3 chapters, Harvard Lecture Notes.

21. C. Năstăcescu and F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, Lecture Notes in Math. **758**, Springer-Verlag, Berlin, 1979.

22. M. Orzech and C. Small, *The Brauer Group of Commutative Rings*, Lecture Notes in Math., Vol. 11, Marcel Dekker Inc., New York, 1975.

23. C. Procesi, *Rings with Polynomial Identities*, Marcel Dekker Inc., New York, 1973.

24. I. Reiner, *Maximal Orders*, Academic Press, London, 1975.

25. J-P. Serre, *Faisceaux Algébriques Coherents*, Ann. Math. **61** (1955), 197–278.

26. J-P. Serre, *Groupes Algébriques et Corps de Classes*, Hermann, Paris, 1959.

27. J-P. Serre, *Corps Locaux*, Hermann, Paris, 1968.

28. B. Stenström, *Rings and Modules of Quotients*, Lecture Notes in Math. **237**, Springer-Verlag, Berlin, 1971.

29. J-P. Van Deuren, J. Van Geel and F. Van Oystaeyen, *Genus and a Riemann–Roch*

Theorem for Non-commutative Function Fields in One Variable, Proc. Sem. Dubreil 1979, Lecture Notes in Math. Springer-Verlag, to appear.

30. J. Van Geel, *Primes and Value Functions*, Ph.D. Thesis, Univ. of Antwerp UIA, 1980.

31. F. Van Oystaeyen, *Generalized Rees rings and arithmetically graded rings*, J. Algebra, to appear.

32. F. Van Oystaeyen, *On Brauer groups of arithmetically graded rings*, to appear.

33. F. Van Oystaeyen, *Graded Azumaya Algebras and Brauer Groups*, *Ring Theory: Antwerp* 1980, Lecture Notes in Math. **825**, Springer-Verlag, Berlin, 1980, pp. 158–171.

34. F. Van Oystaeyen, *On graded rings and modules of quotients*, Comm. Algebra **6** (1978), 1923–1959.

35. F. Van Oystaeyen, *Prime Spectra in Non-commutative Algebra*, Lecture Notes in Math. **444**, Springer-Verlag, Berlin, 1975.

36. F. Van Oystaeyen, *A note on graded P.I. rings*, Bull. Soc. Math. Belg., to appear.

37. F. Van Oystaeyen and A. Verschoren, *Reflectors and Localization, Application to Sheaf Theory*, Lecture Notes in Math., Vol. 41, Marcel Dekker, Inc., New York, 1979.

38. F. Van Oystaeyen and A. Verschoren, *Graded P.I. Rings and Noncommutative Projective Schemes*, Comm. Algebra **9** (1981), 149–178.

39. F. Van Oystaeyen and A. Verschoren, *Introduction to Noncommutative Algebraic Geometry*, Lecture Notes in Math., Springer-Verlag, Berlin, to appear.

40. F. Van Oystaeyen and A. Verschoren, *The geometric interpretation of Brauer groups of graded rings, I*, preprint, U.I.A. University of Antwerp.

41. F. Van Oystaeyen and A. Verschoren, *The relative Brauer group of a graded ring*, to appear.

42. A. Verschoren, *Some Ideas in Noncommutative Algebraic Geometry*, Ph.D. Thesis, University of Antwerp, U.I.A., 1978.

43. A. Verschoren, *On the Picard group of a Grothendieck category*, Comm. Algebra **8** (1980), 1169–1194.

44. A. Verschoren, *The Brauer Group of a quasi-affine scheme*, in *Brauer Groups in Ringtheory and Algebraic Geometry*, Proc. Antwerp 1981 Conference, to appear.

45. O. Zariski and P. Samuel, *Commutative Algebra, I, II*, Van Nostrand Publ. Co., 1960.

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